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Electrostatic capacity of two unequal adhering spheres

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Abstract. We give a detailed general solution of the Dirichlet problem for two exterior touching spheres. From this general result we derive an explicit formula giving the electrostatic capacity of two unequal exterior adhering spheres.

1. Introduction

The calculation of the so called 'plasmon modes' of various cavities imbedded in metals is of growing interest in the modern studies of surface physics. (Ronveaux *et al* 1977, Lucas *et al* 1975). These eigenmodes ω are related to the dielectric constant $\epsilon(\omega)$ by the usual state equation of the metal:

$$\epsilon(\omega) = 1 - \omega_p^2/\omega^2; \qquad \omega_p = (4\pi ne^2/m)^{1/2} \tag{1}$$

e and m are the charge and the mass of the electron, n is the electronic density. The dielectric constant is quantised by the classical boundary condition of electromagnetism at the metal surface. Neglecting the retardation effect, the boundary value problem is:

$$\begin{cases} \nabla^2 V(\boldsymbol{r}) = 0 & \boldsymbol{r} \text{ outside all boundary surfaces} \\ \epsilon \dfrac{\partial V^{\mathrm{OUT}}}{\partial n} = \dfrac{\partial V^{\mathrm{IN}}}{\partial n} & \text{at each boundary.} \end{cases} \tag{2}$$

The dynamical potential solution $V(\boldsymbol{r})\,e^{i\omega t}$ must be always regular, null at infinity and continuous at each surface.

For most of the finite geometry for which the Laplace equation is separable, these modes are known analytically. Our laboratory was recently interested in finding the modes of two adhering metallic spheres in order to compute the exact Van der Waals interaction between two metallic particles. In that case the problem was to solve complicated coupled Fredholm integral equations which have not yet been solved. But the 'static mode', for which the metal surfaces are equipotentials, solves the electrostatic Dirichlet problem for the touching spheres and we realise that the scientific literature apparently does not mention the capacity of these two spheres with respect to infinity. This note derives the following expression for the capacity $C(R_1, R_2)$ of two adhering metallic spheres (see figure 1):

$$C(R_1, R_2) = R_1\{2\psi(1) - \psi[n/(n+1)] - \psi[1/(n+1)]\}[n/(n+1)] \tag{3}$$

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where n is the ratio of the two radii; R_1 and R_2 of the two spheres ($n = R_2/R_1 > 1$) and $\psi(z)$ is the logarithmic derivative of the Euler $\Gamma(z)$ function.

$$\psi(z) = \Gamma'(z)/\Gamma(z)$$

$\psi(1)$ is equal to the Euler constant $\gamma = -0{\cdot}5772$, and the units are such that the capacity C of a unique sphere of radius R is $C = R$.

This formula reduces, when the two spheres are of equal radius R, to $C = 2R \ln 2$ which is a well known result (Lebedev 1965).

2. General solution of the exterior Dirichlet problem for two external adhering spheres

The coordinate system of a tangent sphere (see figure 1) is related to the cartesian coordinates $x\ y\ z$ by the relations (Lebedev 1965, Sneddon 1974).

$$\begin{cases} x = \rho \cos \phi = [a\alpha/(\alpha^2+\beta^2)] \cos \phi \\ y = \rho \sin \phi = [a\alpha/(\alpha^2+\beta^2)] \sin \phi \\ z = -a\beta/(\alpha^2+\beta^2). \end{cases} \tag{4}$$

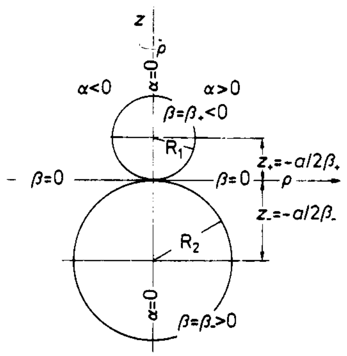

Figure 1. Coordinate system of tangent spheres.

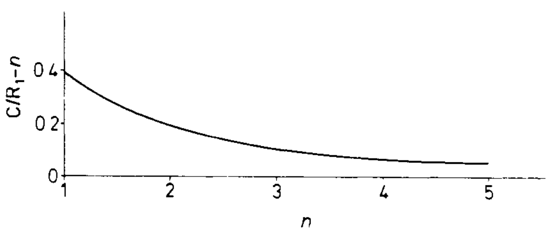

Figure 2. Capacity $C(R_1R_2)$ of the two adhering spheres ($n = R_1/R_2 > 1$).

The Laplace equation $\nabla^2\psi(\alpha, \beta, \phi) = 0$ is only R separable in this orthogonal system of coordinates via the change of function:

$$\psi(\alpha, \beta, \phi) = (\alpha^2+\beta^2)^{1/2} A(\alpha) B(\beta). \tag{5}$$

The function $A(\alpha)$ is a solution of the Bessel equation:

$$\frac{d^2A}{d\alpha^2} + \frac{1}{\alpha}\frac{dA}{d\alpha} + \left(\mu^2 - \frac{m^2}{\alpha^2}\right)A = 0 \tag{6}$$

and the function $B(\beta)$ is a solution of the constant coefficient equation

$$\frac{d^2B}{d\beta^2} - \mu^2 B = 0 \tag{7}$$

where μ and m are the constants of separation:

$$\begin{cases} m = 0, 1, 2, \ldots \\ 0 < \mu < \infty. \end{cases} \tag{8}$$

The solution of the exterior Dirichlet problem for which the potential reduces respectively to $V_1(\alpha, \phi)$ and $V_2(\alpha, \phi)$ on the sphere $\beta = \beta_+$ and $\beta = \beta_-$ is given by superposition of the elementary solution and is:

$$\underset{\text{ext}}{V}(\alpha, \beta, \phi) = \tfrac{1}{2}(\alpha^2+\beta^2)^{1/2} \sum_{m=0}^{\infty} \delta_m[a_m \cos m\phi + b_m \sin m\phi]$$
$$\times \int_0^{\infty} \left(N(\mu) \frac{\sinh \mu(\beta - \beta_+)}{\sinh \mu(\beta_- \beta_+)} + M(\mu) \frac{\sinh \mu(\beta - \beta_-)}{\sinh \mu(\beta_- + \beta_+)} \right) \mu J_m(\alpha\mu)\, d\mu$$

with

$$\begin{cases} \delta_0 = 1 \\ \delta_m = 2 \qquad m = 1, 2, \ldots. \end{cases} \tag{9}$$

The advantage of this representation lies in the fact that $M(\mu)$ depends only on V_1 and $N(\mu)$ on V_2 via the integral equation obtained from the usual Fourier decomposition of $V_j(\alpha, \phi)$ in components $C_{\pm}^{(m)}(\alpha)$ and $S_{\pm}^{(m)}(\alpha)$:

$$V_j(\alpha, \phi) = \tfrac{1}{2} \sum_{m=0}^{\infty} \delta_m\{C_{\pm}^{(m)}(\alpha) \cos m\phi + S_{\pm}^{(m)}(\alpha) \sin m\phi\}$$
$$\times (\alpha^2 + \beta_{\pm}^2)^{1/2} \int_0^{\infty} \begin{Bmatrix} a_m M(\mu) \\ b_m N(\mu) \end{Bmatrix} J_m(\alpha\mu)\mu\, d\mu$$
$$= \frac{1}{\pi} \int_{-\pi}^{+\pi} V_j(\alpha, \phi) \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} d\phi = \begin{Bmatrix} C_{\pm}^{(m)}(\alpha) \\ S_{\pm}^{(m)}(\alpha) \end{Bmatrix} \qquad (j = 1, 2). \tag{10}$$

The functions $a_m N(\mu)$ and $b_m M(\mu)$ are now defined by their Hankel transforms

$$(\alpha^2 + \beta_{\pm}^2)^{-1/2} \begin{pmatrix} C_{\pm}^{(m)}(\alpha) \\ S_{\pm}^{(m)}(\alpha) \end{pmatrix}$$

and the potential at any (α, β, ϕ) is now given, after permutation of the two integrals, by:

$$V(\alpha, \beta, \phi) = (2\pi)^{-1} \sum_{m=0}^{\infty} (\mathrm{A}_m(\alpha, \beta) \cos m\phi + \mathrm{B}_m(\alpha, \beta) \sin m\phi)(\alpha^2 + \beta^2)^{1/2}$$

with:

$$\begin{Bmatrix} A_m(\alpha, \beta) \\ B_m(\alpha, \beta) \end{Bmatrix} = \delta_m \int_0^{\infty} d\mu\mu\, J_m(\alpha\mu) \exp[\pm\mu(\beta - \beta_{\pm})] \int_0^{\infty} \frac{\lambda J_m(\lambda\mu)}{(\lambda^2 + \beta_{\pm}^2)^{1/2}} \begin{Bmatrix} C_{\pm}^{(m)}(\lambda) \\ S_{\pm}^{(m)}(\lambda) \end{Bmatrix} d\lambda$$

at any *interior* point of the β_+ and β_- spheres, and at any *exterior* point:

$$\begin{Bmatrix} \mathrm{A}_m(\alpha, \beta) \\ \mathrm{B}_m(\alpha, \beta) \end{Bmatrix} = \delta_m \int_0^{\infty} \frac{\mu J_m(\alpha\mu)\, d\mu}{\sinh \mu(\beta_- - \beta_+)} \left(\sinh \mu(\beta - \beta_+) \int_0^{\infty} \frac{\lambda J_m(\lambda\mu)}{(\lambda^2 + \beta_-^2)^{1/2}} \begin{Bmatrix} C_-^{(m)}(\lambda) \\ S_-^{(m)}(\lambda) \end{Bmatrix} d\lambda \right.$$
$$\left. - \sinh \mu(\beta - \beta_-) \int_0^{\infty} \frac{\lambda J_m(\lambda\mu)}{(\lambda^2 + \beta_+^2)^{1/2}} \begin{Bmatrix} C_+^{(m)}(\lambda) \\ S_+^{(m)}(\lambda) \end{Bmatrix} d\lambda \right). \tag{11}$$

In the next section, the Fourier components $C_\pm$ and $S_\pm$ are constant functions for $m=0$ and the last integrals can be exactly computed via the well known result:

$$\int_0^\infty \frac{\lambda J_0(\lambda\mu)\,d\lambda}{(\lambda^2+\beta_\pm^2)^{1/2}} = \frac{\exp(-|\beta_\pm|\mu)}{\mu}. \tag{12}$$

We also need the Hankel converse integral:

$$\int_0^\infty J_0(\alpha\mu)\exp(-\mu|\beta_\pm|)\,d\mu = (\alpha^2+|\beta_\pm^2|)^{-1/2} \tag{13}$$

3. Capacity of the two touching spheres

If the given potentials $V_j(\alpha,\phi)$ reduce to the constant V_0, the charge density σ_j on each of the two conducting spheres β_+ and β_- are given by the normal derivative of the following potential ($j=1,2$):

$$V(\alpha,\beta) = V_0(\alpha^2+\beta^2)^{1/2}\int_0^\infty \frac{J_0(\alpha\mu)\,d\mu}{\sinh\mu(\beta_- - \beta_+)}\{\sinh\mu(\beta-\beta_+)\exp(-\mu|\beta_-|)$$
$$-\sinh\mu(\beta-\beta_-)\exp(-\mu|\beta_+|)\} \qquad (m=0). \tag{14}$$

The charge density $\sigma_j(\alpha)$ and the total charge Q_j on each sphere are computed using the definition

$$\begin{cases} \sigma_j = \mp({}^h\beta_\pm)^{-1}\left.\dfrac{\partial V_j}{\partial\beta}\right|_{\beta=\beta_\pm} \\ Q_j = \displaystyle\iint \sigma_j(\alpha)\,dS_\pm = 2\pi a^2\int_0^\infty \frac{\sigma_j(\alpha)\alpha\,d\alpha}{(\alpha^2+|\beta_\pm^2|)^2} \qquad (j=1,2) \end{cases}$$

and

$${}^h\beta_\pm = a/(\alpha^2+\beta_\pm^2) \tag{15}$$

In the following, the index $j=1$ ($j=2$) corresponds to the $+$ ($-$) sign in all indexed quantities.

Each integral in the β derivative of equation (14) is of the type (12) or (13) or obtained from these integrals by differentiation with respect to β. The charge density σ_j can therefore be written as:

$$\sigma_j = \frac{V_0}{a}\Bigg(|\beta_\pm| + (\alpha^2+|\beta_\pm|^2)^{3/2}$$
$$\times\int_0^\infty \frac{J_0(\alpha\mu)\mu\,d\mu\,[\exp(-\mu|\beta_\pm|)\cosh\mu(|\beta_+|+|\beta_-|) - \exp(-\mu\beta_\mp)]}{\sinh\mu(|\beta_+|+|\beta_-|)}\Bigg). \tag{16}$$

The total charge $Q = Q_1+Q_2$ is now

$$Q = \tfrac{1}{4}aV_0[|\beta_+|^{-1}+|\beta_-|^{-1}] + \tfrac{1}{2}aV_0 I(|\beta_+|,|\beta_-|) \tag{17}$$

with

$$I=\int_0^\infty \mathrm{d}\mu \frac{[\exp(-\mu|\beta_+|)\cosh\mu(|\beta_+|+|\beta_-|)-\exp(-\mu|\beta_-|)]\exp(-\mu|\beta_+|) + [\exp(-\mu|\beta_-|)\cosh\mu(|\beta_+|+|\beta_-|)-\exp(-\mu|\beta_+|)]\exp(-\mu|\beta_-|)}{\sinh\mu(|\beta_+|+|\beta_-|)}. \tag{18}$$

Now the change of variable $x=a\mu$ and the relation between each radius and the β, $|\beta_\pm|=a/2R_i$, gives:

$$C=\frac{1}{2}\left\{(R_1+R_2) + \int_0^\infty \mathrm{d}x \frac{\cosh((R_1+R_2)x/2R_1R_2)\,(\exp(-x/R_1)+\exp(-x/R_2)) - 2\exp[-x((R_1+R_2)/2R_1R_2)]}{\sinh((R_1+R_2)x/2R_1R_2)}\right\}. \tag{19}$$

If we now introduce the ratio n of the radii ($n=R_1/R_2$), the capacity can be written:

$$C=R_1\left[\frac{1}{2}(1+n)+\frac{2n}{1+n}(2\ln 2-1)+\frac{n}{n+1}J(n)\right] \qquad (n\geqslant 1)$$

with

$$J(n)=\int_0^\infty \mathrm{d}y\coth y\left[\exp\left(-\frac{2n}{n+1}y\right)-2\exp(-y)+\exp\left(-\frac{2}{n+1}y\right)\right]. \tag{20}$$

The negative signs in the exponentials ensure the convergence of the integral at infinity. At the origin, the bracket inside the integral cancelled the singularity of the coth function.

Explicitly the three integrals behave as (Gradshteyn and Ryzhik 1965):

$$\int_0^\infty \coth y\exp(-\beta y)\,\mathrm{d}y=\lim_{z\to 1}(z-1)^{-1}-\psi(\beta/2)-\beta^{-1}, \tag{21}$$

where the ψ function is the usual logarithmic derivative of the Γ function. After subtraction of the singularity the capacity takes the form indicated in the introduction.

In order to use tables of the $\psi(z)$ function, usually given for $z>1$, we transform the capacity formula in the following form:

$$C(R_1R_2)=R_1\frac{n}{n+1}\left(2\psi(1)-\psi\left(\frac{2n+1}{n+1}\right)-\psi\left(\frac{n+2}{n+1}\right)+n+2+\frac{1}{n}\right). \tag{22}$$

The reflection formula (Abramowitz and Stegun 1965):

$$\psi(1-z)=\psi(z)+\pi\cot\pi z \tag{23}$$

allows us to write the capacity formula with only one ψ function.

4. Conclusions

The capacity formula (equation (22)) shows that for large n the capacity reduces to $nR_1=R_2$, which is obvious from a physical point of view. R_1 is the larger sphere.

The asymptotic domain is reached very quickly. We have therefore plotted in figure 2 the graph of $C/R_1 - n$ in terms of n in the interval $1 \leqslant n \leqslant 5$.

The twin adhering conducting sphere is used as absolute instrument for measuring high voltage by frequency measurement of the system oscillating in the field (Smith and Rungis 1975, Love 1975).

The sphere geometry is easier to manufacture than the ellipsoidal geometry which was used before.

The induced dipoles due to the oscillations can be computed exactly for an ellipsoid and also for the twin sphere geometry with two equal spheres. (Smith and Rungis 1975). From our developments it would not be difficult to solve exactly the induced dipole problem in the more general case of two spheres of different radii.

Note that in figure 2 we restrict the domain of variation of the ratio n to the range $1 \leqslant n < \infty$, or because, for obvious reasons, the range $0 < n \leqslant 1$ is reached from the range $1 \leqslant n < \infty$ by permutation of the two spheres.

References

Abramowitz, M and Stegun I A 1970 *Handbook of mathematical functions with graphs and mathematical tables* (New York: Dover)

Gradshteyn I S and Ryshik I M 1965 *Table of integrals series and products* (New York and London: Academic)

Lebedev N N 1965 *Special Functions and Applications* (New York: Prentice-Hall)

Love J D 1975 *J. Phys. E: Sci. Instrum.* **8** 1063

Lucas A A, Ronveaux A, Schmeits M and Delanaye F 1975 *Phys. Rev.* B **12** 5372.

Smith W E and Rungis J 1975 *J. Phys. E: Sci. Instrum.* **8** 379–82

Ronveaux A, Moussiaux A and Lucas A A 1977 *Can. J. Phys.* **55**, **16** 1407–22

Sneddon I N 1974 *The use of Integral Transforms, Tata, New Delhi* (New York: McGraw-Hill)